Extending Equilibria to Periodic Orbits for Walkers using Continuation Methods

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***Abstract*— We present a strategy for generating period-one, open-loop walking gaits for multi-degree-of-freedom, planar biped walkers. Our approach uses equilibria of the dynamics as templates, which we connect to a family of period-one walking motions using numerical continuation methods. We define a gait as a fixed point of the walker's hybrid dynamics which resides in a state-time-control space consisting of the robot's post-impact state, switching time (the time at which the swing leg impacts the ground), and a finite set of design or control parameters.**

We demonstrate our approach on several physically-symmetric biped walkers. In particular, we prove that our approach reduces the search space for an initial gait in the state-time-control space to a one-dimensional search in switching time. We show that we can generates periodic motion without resorting to splines or reference trajectories. Finally, we compare our method to generating gaits with virtual holonomic constraints.

I. Introduction

The automatic generation of walking gaits for bipedal robots has been a challenging problem in the robotics literature [1], [2]. While the literature on the subject dates back to over three decades worth of work filled with impressive advancements, there is still no automated method for generating gaits for multi-degree-of-freedom bipeds that takes advantage of the walker's "natural" dynamics, namely the use of gravitational forces to help propel the swing leg during a step. In this paper, we take a step toward solving this problem for physically-symmetric, planar bipeds.

In our work, we consider gaits that reside in a state-time-control space consisting of the robot's post-impact state, a switching time (the time of impact between the swing leg and the ground), and a set of design or control parameters (e.g. the physical parameters of a mechanical system, coefficients of a forcing function, etc.). In this state-time-control space, we define a gait as a fixed point of the walker's hybrid dynamics. The hybrid system is partitioned into a single support phase and an instantaneous double support phase. During the single support phase, the robot pivots about its stance leg while the swing leg is able to pivot freely about the hip. After an impact the role of the two legs are reversed.

A major challenge in automating the process of generating gaits is finding the right combination of state variables and design parameters that will yield a periodic walking gait.

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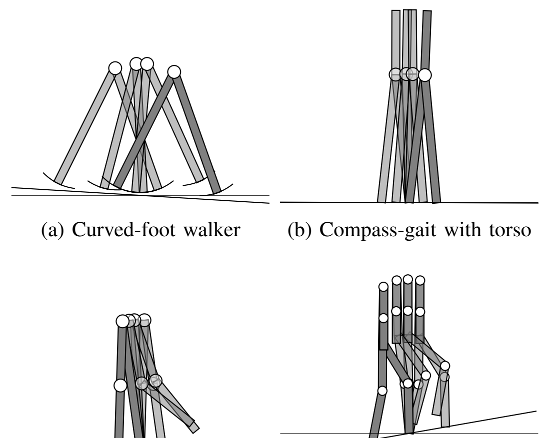


(a) Curved-foot walker (b) Compass-gait with torso

(c) Point-foot kneed walker (d) Humanoid walker

Fig. 1. Example of period-one gaits for various planar biped walkers. The dots are joint centers. All gaits are walking passively downhill. The gaits in (a) and (b) are walking from left to right, while (c) and (d) are walking from right to left.

For many gait generation algorithms, the user has to input a guess for these values. The algorithm then refines the guess using standard optimization or root-finding techniques. An algorithm that uses a guess as an input has several drawbacks, including convergence issues if the guess is not "close enough" to an actual gait and provides no guidelines for making a good guess as the degrees of freedom and the number of parameter are increased.

In our work, we eliminate this guesswork by using continuation methods. Continuation methods are useful algorithms for exploring the behavior of the system as its parameters are varied. Within the walking literature, there are several works where the parameters of a biped model are varied in order to explore the effects of a parameter on a gait ([1], [3], [4]). In each of these works, we see fixed points forming connected components in their respective state-time-control space.

By itself, a continuation method suffers from an even worse version of the guessing problem as it needs an actual gait before it can find more. We get around this restriction by using equilibria that are also fixed points as templates and use continuation methods to "trace" their connected component until we find a branch of fixed points with walking gaits. When the control parameters are held constant, this approach reduces the search space for a gait to a 1-D search problem in switching time. Figure 1 shows a diverse set of walking models where we started with a "standing still" equilibrium

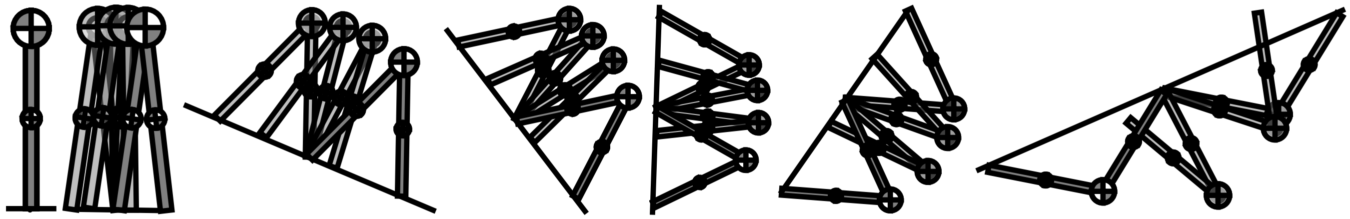


Fig. 2. Passive gaits in the same connected component as the compass-gait walker's equilibrium point (left-most cartoon) found with numerical continuation methods. The final three gaits resemble overhand brachiation. The gaits above the ground are walking from left to right, below the slope from right to left, and the walker on the vertical slope is traveling downwards.

point and successfully extended the template to a walking gait.

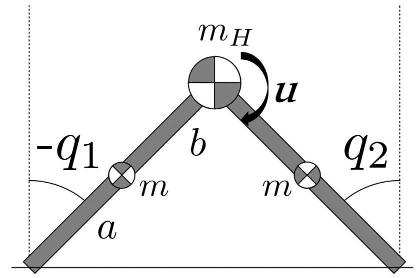
A. Statement of Contributions

We show how a few topological and differential geometric concepts can significantly simplify the gait generation problem avoiding the use of model reduction or splines to generate gaits. Our contributions are

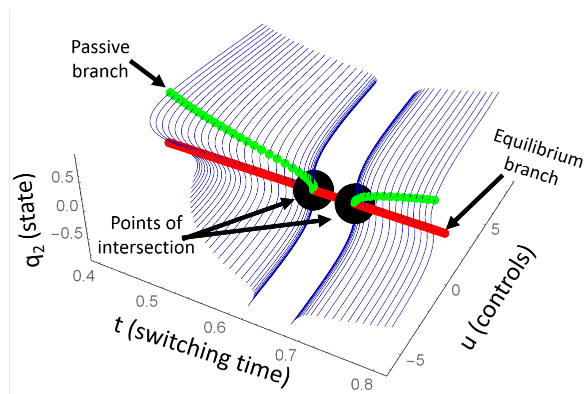
- **An easy and reliable way to generate gaits from an equilibrium point.** We show how an equilibrium point (our template motion) reduces the search for an initial moving gait to a one-dimensional search in time. In other words, the search space is independent of the degrees of freedom and the physical parameters of the system.
- **A geometric interpretation of periodic gaits as points on a manifold.** We transform the gait generation problem from a two-point boundary value problem solved by optimization techniques to generating a manifold of periodic walking motions embedded in a state-time-control space using continuation methods.
- **The ability to generate open-loop gaits based on a finite set of design or control parameters.** We show how a finite vector consisting of n state variables and k design or control variables can be used to generate open-loop gaits. As an example, we place a torsional spring and motor at the hip of a three-link model (Figure 1(b)) as in [5]. The controls are then parameterized by a spring constant and a constant torque ($k = 2$).
- **A comparison with another gait generating algorithm.** We compare our approach to generating gaits with virtual holonomic constraints addressing strengths and weaknesses in both techniques.

B. A motivating example

Figures 2 and 3 illustrate our approach using the compass-gait biped. In this model, we assume that there is an actuator creating a constant torque at the hip joint, thus creating a 6-D state-time-control space consisting of the 4 states, a switching time, and a constant torque value. In Figure 2, the "standing upright" equilibrium point of the compass-gait is successfully extended from a walking gait to overhand brachiation over a range of extreme slopes. While the cartoon animations give a visual idea of the robot's motion, each gait can also be represented as a point in a 6-D state-time-control space as shown in Figure 3(b). This figure shows a small subset of the connected component of our template projected onto a 3-D subspace. While the equilibrium point would form a straight line through the space connecting the two black dots, it eventually intersects with a set of passive solutions (denoted in green). The points of intersection correspond to the "short" and "long" solutions reported in [1]. These passive gaits themselves are a part of an even larger space of powered gaits. Using a counting argument, we expect the dimension of the manifold to equal the number of control variables plus the switching time as we have $n = 4$



(a) The compass-gait model



(b) 3-D projection of the state-time-control subspace

Fig. 3. A projection of a 6-D state-time-control space for (a) the compass gait model onto (b) the impact time, stance-leg angle, and constant torque dimensions. Each point on the surface is a fixed point. The large black dots are where the branch of equilibrium points (red line) of the compass-gait intersects with the curve of passive walking gaits (green) where $u = 0$. A few of these passive gaits are shown in Figure 2. The other gaits (in blue) are powered at the hip joint. The challenge is finding the two switching times denoted by the black dots that allow us to push off the equilibrium branch and onto the branch that contains walking gaits.

constraints (the final post-impact state must equal the initial post-impact state) and $n + k + 1 = 6$ freedoms yielding a $(k + 1)$-dimensional set of solutions.

The main challenge is knowing when the branch of equilibrium points intersects with the branch of walking gaits. We outline a method for doing so in Section III.

A video presentation of this example with animations of the gaits can be found in the accompanying video attachment.

C. Previous Work

There are several works that present techniques for generating gaits using optimization techniques [6], [7], [8], [9], [10], [11], [12]. Optimization techniques can be used to find gaits for an arbitrary biped model, but applying standard optimization techniques does not take advantage of the connected components of a walking gait [13]. While flexible, these approaches are limited by how good the initial guess is which varies from model to model.

The idea of using templates to mitigate the issues with optimization techniques is not new [14], [15], [16], [17]. A key observation in this line of work is that walking is a highly coordinated motion that does not require the full degrees of freedom to accomplish. Many techniques that take advantage of this insight focus on using reduced models of lower dimension to generate gaits. The complicated task of making a high degree-of-freedom biped walk is then equivalent to making a lower-dimensional version of the biped (e.g., knees and elbows are locked during a portion of the motion) perform the same task through, for example, holonomic constraints [18], model linearization, or a library of motion primitives [19].

Arguably the most successful walking algorithms in the field utilize the zero moment point (ZMP) [20] as an important component in gait generation. The ZMP is defined as the point where the net horizontal moment is zero and can be seen as an extreme case of model reduction based on templates. The ZMP applies to any arbitrary robot model. There are drawbacks to using the ZMP approach, such as flat-footed walking, high cost-of-transport relative to human walking, and quasi-static dynamics (which leads to poor agility and reaction time).

Our work can be viewed as a generalization of the previous work on simple walking models ([4], [1], [3]). These works also search for gaits near an equilibrium point of their respective model and have yielded invaluable insight into the mechanics of walking. This body of work uses continuation methods to explore the effects of various parameters, such as ground slope, on the biped's gait. For generating motion sequences for animated individuals, an impressive application of continuation methods for generating parkour motions based on a human motion database can be found in [21]. We complement their work by providing further insight into why continuation methods can be an effective tool in generating gaits even without a library of motion primitives.

D. Paper Outline

We begin in Section II with the problem statement as well as background information that will be used throughout the paper. In Section III, we show how an equilibrium point is extended to a periodic walking gait. Section IV presents an example of our techniques on various models of symmetric biped walkers. We then conclude and discuss future work in Section V.

II. Problem Statement

We are interested in finding period-one fixed points of a multi-degree-of-freedom robot with continuous swing-leg dynamics during the single support phase of a step and an instantaneous impact with the ground during the double support phase. The robot has dynamics of the form

$$\begin{aligned} \dot{x}(s) &= F(x(s), u(s), \lambda) \quad & 0 < s < t \\ x^+ &= H(x(s), \lambda) \quad & s = t, \end{aligned} \tag{1}$$

where t is the switching time when the impact occurs, $\lambda \in \mathbb{R}^k$ is a vector of design or control variables, $x \in \mathbb{R}^n$ is the state of the system, $x^+ \in \mathbb{R}^n$ is the state after a step, $u \in \mathbb{R}^p$ is the forcing input, $F\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the swing-leg dynamics and $H\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is the jump map which maps the pre-impact state to the post-impact state.

A walking gait requires that the post-impact state x_0 at the beginning of a biped's step equal the post-impact state x^+ at the beginning of the robot's next step. This step-to-step map can be written as a mapping $G\colon \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^n$ such that

$$x^+ = G(x_0, \lambda, t) = H(x(x_0, \lambda, t), \lambda), \tag{2}$$

where $x(x_0, \lambda, t)$ is the state at time t satisfying $\dot{x} = F(x, u, \lambda)$ starting from the post-impact state x_0.

A period-one fixed point is a point $(x_0^*, \lambda^*, t^*) \in \mathbb{R}^{n+k+1}$ such that $x_0^* = G(x_0^*, \lambda^*, t^*)$ and it resides in an $(n+k+1)$-dimensional state-time-control space. The state corresponds to the robot's post-impact state at the beginning of a new step and the time and controls correspond to the switching time and design parameters, respectively.

If the map G is continuously differentiable, then the set of fixed points

$$X = \{(x_0^*, \lambda^*, t^*)\colon G(x_0^*, \lambda^*, t^*) - x_0^* = 0\}$$

will have additional local structure forming $(k + 1)$-dimensional manifolds [22]. Neighboring points in X will locally form a manifold whenever the Jacobian

$$J(x_0, \lambda, t) = \left[\tfrac{\partial G}{\partial x_0} - I, \quad \tfrac{\partial G}{\partial \lambda}, \quad \tfrac{\partial G}{\partial t}\right], \tag{3}$$

where I is the $n \times n$ identity matrix, has maximal rank n. This follows from the implicit function theorem [22].

We will refer to fixed points $(x_0^*, \lambda^*, t^*) \in X$ with $\operatorname{rank}(J(x_0^*, \lambda^*, t^*)) = n$ as *regular* points of G. Fixed points are called *singular* if they are not regular [23].

Regular points and singular points will play a central role in transforming an equilibrium point into a walking gait. Referring back to Figure 3, the connected component of the compass-gait walker's equilibrium point is comprised of several manifolds joined together at singular points. If we confine ourselves to just the state-time subspace as shown in

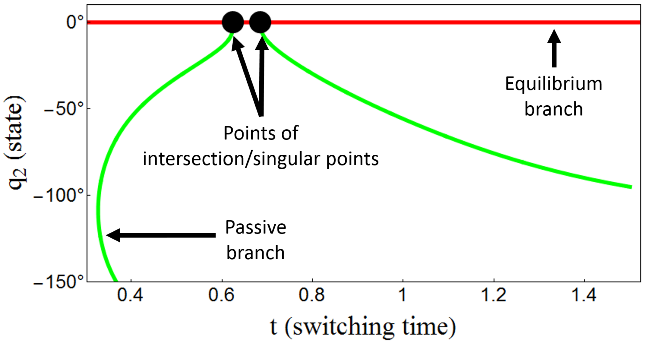


Fig. 4. A 2-D slice of Figure 3(b) at $u = 0$. This slice shows a projection onto the passive subspace of the state-time-control space of the compass-gait walker. The "points of intersection" (black dots) are singular points. These points allow us to switch from the equilibrium branch (red) and onto a branch with walking gaits (green).

Figure 4, the singular points are the two black dots. These singular points allow us to switch between different branches of fixed points on the connected component. We can generate the connected component by using continuation methods.

To summarize, given

- a hybrid dynamical system $\mathcal{H} = (F, H)$ of the form (1),
- a vector of control parameters $\lambda \in \mathbb{R}^k$,
- and a continuously differentiable map G satisfying (2),

use numerical continuation methods to find solutions to

$$G(x_0, \lambda, t) - x_0 = 0$$

starting from a set of isolated equilibria satisfying $(x^*_{\text{eq}}, \lambda^*, t^*) \in X$ and $F(x^*_{\text{eq}}, u^*, \lambda^*) = 0$.

III. Generating periodic walking gaits in the state-time subspace

We now briefly review how continuation methods work and then proceed to show how our template motion, an equilibrium point, can be extended to a walking gait with a nonzero step size in the state-time subspace.

A. Numerical Continuation Methods

Continuation methods are useful numerical tools for exploring bifurcations and generating implicitly defined manifolds $M(x, y) = 0$. For us, these manifolds will consist of fixed points of our step-to-step map G. While multi-parameter continuation methods exist [24], we will focus on generating implicitly defined curves in an $(n + 1)$-dimensional subspace which will be the case later on in this section. This assumption is not too restrictive as we can fix k variables in our state-time-control space and apply a numerical continuation method in the $(n + 1)$-dimensional subspace. This is what we did to generate the surface in Figure 3(b).

The premise behind numerical continuation methods is that given a point $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, a curve $c\colon \mathbb{R} \to \mathbb{R}^{n+1}$ that goes through it, say at $c(0)$, and a continuously differentiable map $M\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$, the solution to $M(c(0)) = 0$ has geometric information that will allow us to find the next point on the curve such that $M(c(s)) = 0$. If we differentiate M with respect to its inputs and evaluate at $s = 0$

$$\frac{\partial M}{\partial c}(c(0))\dot{c}(0) = 0,$$

then we can obtain the tangent space at $c(0)$ by solving for the null space of $\frac{\partial M}{\partial c}(c(0)) \in \mathbb{R}^{n\times(n+1)}$. The tangent to the curve will then be a linear combination of the vectors in the tangent space. Furthermore, if $c(0)$ is a regular point, then the implicit function theorem tells us that there will only be one tangent vector at the point $c(0)$ and that there exists a small neighborhood of fixed points near $c(0)$ that all lie on the same curve. In order to be able to switch to a different branch of the connected component, we require a singular point.

Algorithm 1 Euler-Newton predictor-corrector algorithm

Require: c_0 such that $G(c_0) = 0$ and a step size h.
Initialize the array of fixed points c.
$c[0] = c_0$
for $i := 1..N$ **do**
  Prediction Step:
  Compute $\dot{c}$ by solving $\frac{\partial G}{\partial c}(c[i-1])\dot{c} = 0$ and $||\dot{c}|| = 1$.
  $z = c[i-1] + \dot{c}h$
  Correction Step:
  Solve for $H(z) = 0$ and $\dot{c}^T(z - c[i-1]) = h$ using Newton's method.
  $c[i] = z$
end for
return the solution curve c.

The algorithm used to generate the gaits in this paper is a predictor-corrector algorithm known as the Euler-Newton algorithm [23]. Algorithm 1 is a barebones version of the algorithm. Geometrically, this algorithm defines a hyperplane a distance h away from the current solution where the search for the next point on the curve takes place; the hyperplane is normal to the tangent of a known solution. Next, a prediction step selects a point on the plane as the initial guess and a root finding method iteratively refines the guess until the point is again back on the curve.

B. Extending equilibrium points in the state-time subspace with parameters held constant ($n > k = 0$)

A major drawback with continuation methods is that they are only useful once a point on the curve is known. We show how equilibrium points can be used to bootstrap the process and can be extended to periodic motions with a net displacement whenever $(x^*_{\text{eq}}, \lambda^*, t^*)$ is a singular point. Our approach relies on the following assumptions:

- EH1 Equilibria are isolated from each other.
- EH2 The size of the k-dimensional design parameter space pertaining to λ is zero, so that a fixed point is fully specified by the (x_0, t) space.
- EH3 There is only one impact event and it occurs at time $t > 0$.

EH4 There exists an equilibrium point $(x^*_{\text{eq}}, t^*) \in X$ such that $F(x^*_{\text{eq}}, u(t^*)) = 0$ for all t^* with walking gaits with nonzero step length in its connected component.

EH5 The forcing function and the initial jump are independent of t, so that $\frac{\partial u}{\partial t}, \frac{\partial x_0}{\partial t} = 0$.

Equilibrium hypotheses EH1, EH2, and EH3 are meant to avoid special cases. For example, we require $k = 0$ in EH2 to avoid branches that might only contain gaits with zero step sizes. As a simple case, consider the compass gait at an equilibrium point in X. If, in addition, the link's length was a design parameter, then a useless branch where the length grows without bound exists at every time step while still remaining at the equilibrium. A similar reasoning is behind EH1, but in this case the useless branch will be a different equilibrium point.

Since we assume that $k = 0$, we will refer to the reduced state-time space (x_0, t) with an appropriately defined Jacobian $J(x_0, t) = \left[\frac{\partial G}{\partial x_0} - I, \quad \frac{\partial G}{\partial t}\right]$.

EH3 eliminates multiple time-based impact events such as knee strike commonly found with kneed walkers. Our results can be extended to handle multiple switching times (Figure 1(c) is an example), but because of space constraints we only pursue one switching time.

One limitation of our analysis is that we cannot prove that there always exists an equilibrium point that will connect to a set of walking gaits. Instead, we take it as an assumption (EH4) and prove that if such an equilibrium exists, our approach will theoretically find walking gaits without fail.

In the remainder of this section, we prove that

- equilibria that satisfy EH4 have a connected component of nonzero length,
- singular points occur only when $\frac{\partial G}{\partial x_0}$ is not full rank,
- and our search for an initial gait is always one-dimensional regardless of the number of states.

We now show that equilibria have a connected component of nonzero length.

Proposition 1: If $F(x^*_{\text{eq}}, u(s)) = 0$ for all time s, then $\frac{\partial G}{\partial t} = 0$.

Proof: Differentiating G with respect to t and using $\frac{\partial u}{\partial t} = \frac{\partial x_0}{\partial t} = 0$ from EH 5, we get

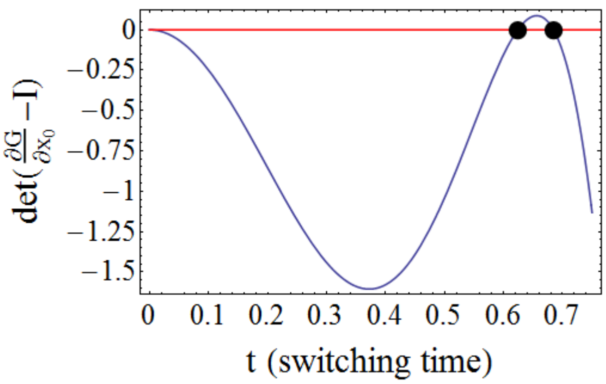
$$\frac{\partial G}{\partial t}(x^*_{\text{eq}}, t^*) = \frac{\partial H}{\partial x} \circ \left(\frac{\partial x}{\partial x_0}\frac{\partial x_0}{\partial t} + \frac{\partial x}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial x}{\partial t}\right) \circ (x^*_{\text{eq}}, t^*)$$
$$= \frac{\partial H}{\partial x} F(x^*_{\text{eq}}, u(t^*)) = 0$$

■

Corollary 1: If $(x^*_{\text{eq}}, t^*) \in X$ is a period-one fixed point such that x^*_{eq} is an equilibrium point of F, then there will always exist a "trivial" branch of fixed points (x^*_{eq}, t^*) for all values of $t^* \in \mathbb{R}$.

This result follows from Proposition 1 as the null space of $J(x^*_{\text{eq}}, t^*)$ will always have a vector that points in the switching-time coordinate direction.

If (x^*_{eq}, t^*) is a regular point, then the trivial branch will be the only branch of fixed points in a small neighborhood of (x^*_{eq}, t^*). Singular points do not have this restriction.



(a)

Fig. 5. The $\det(\frac{\partial G}{\partial x_0} - I)$ for the compass-gait walker. The roots of the function correspond to the smallest nonzero switching times where the connected component branches off of the equilibrium branch of gaits and onto a passive branch of gaits.

In order for (x^*_{eq}, t^*) to be a singular point, we need $\operatorname{rank}(J(x^*_{\text{eq}}, t^*)) < n$. From inspection of J this means that

Corollary 2: (x^*_{eq}, t^*) is a singular point if and only if $\frac{\partial G}{\partial x_0}$ has an eigenvalue of 1.

It is very easy to compute whether or not a singular point exists within a certain interval $t \in [a, b]$ by simply solving $\det\left(\frac{\partial G}{\partial x_0} - I\right) = 0$. As a special case, if the linearized dynamics of (1) has the form $\dot{\Phi}(s) = A\,\Phi(s)$ where $\Phi = \frac{\partial x}{\partial x_0}$ is the linearized dynamics about the equilibrium point and $A = \frac{\partial F}{\partial x}(x_{\text{eq}}, u)$ is a constant $n \times n$ matrix, then singular points occur whenever

$$\det\left(\frac{\partial H}{\partial x} e^{At} - I\right) = 0, \tag{4}$$

where e^{At} is the matrix exponential. The only parameter that varies in this equation is t.

We have now reduced the problem of finding an initial gait from a potentially $(n + k + 1)$-dimensional search problem to a one-dimensional search problem that is independent of the number of states n and the number of parameters k. If the connected component of an equilibrium point contains any walking gaits, we can find the corresponding branch by searching for singular points on the trivial branch of fixed points. This is equivalent to finding the root of $\det(\frac{\partial G}{\partial x_0}(x^*_{\text{eq}}, t))$. We can easily visualize the zeros of this function as a function of t (see Figure 5).

IV. Examples

A. Revisiting the compass-gait, two-link model

As a first example of our method, we start with the compass-gait walker, which has $n = 4$ states. The details of the model can be found in [1]. The equilibrium point we are interested in is the upright, folder over configuration shown in the left-most animation of Figure 2. The paper explores passive gaits, so there are no control or design parameters ($k = 0$). We search for the first two singular points with a switching time greater than zero. This requires solving for the roots of (4) as shown in Figure 5. The matrices of interest for solving (4) are

$$\frac{\partial F}{\partial x}(x^*_{\text{eq}}, \lambda^*) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g(\beta^2(\mu+1)+2\beta(\mu+1)+\mu+2)}{\alpha\beta((\beta+1)^2\mu+1)} & \frac{(\beta+1)g(\beta\mu+\beta+\mu+2)}{\alpha\beta((\beta+1)^2\mu+1)} & 0 & 0 \\ -\frac{(\beta+1)g}{\alpha((\beta+1)^2\mu+1)} & \frac{g(\beta\mu+\beta+\mu+2)}{\alpha((\beta+1)^2\mu+1)} & 0 & 0 \end{bmatrix}$$

$$\frac{\partial H}{\partial x}(x^*_{\text{eq}}, \lambda^*) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\beta+1}{\mu(\beta+1)^2+1} & \frac{\mu\beta^2+2\mu\beta+\beta+\mu+2}{\mu(\beta+1)^2+1} \\ 0 & 0 & -\frac{\beta}{\mu(\beta+1)^2+1} & \frac{(\beta+1)(\beta\mu+\mu+1)}{\mu(\beta+1)^2+1} \end{bmatrix}$$

Solving for these roots using the common parameters of $g = 9.81$, $\alpha = 0.5$, $\beta = \frac{b}{a} = 1$, and $\mu = \frac{m_H}{m} = 2$ yields switching times of $t \approx 0.62$ s and $t \approx 0.68$ s (see Figure 5).

A similar analysis has been done before on the compass gait, but to our knowledge we are the first to use a geometric framework that scales without any dependence on the degrees of freedom. As long as there is only one switching time, nothing about our framework changes as n is increased. The same steps are repeated for finding gaits of other walking models. After finding the two singular points, we can apply Algorithm 1 to trace the connected component from these branch points. The results of the continuation method can be seen earlier in Figures 2 and 3.

B. *A more complicated model*

We have also tested our technique on a nine-link planar model which has a 19-dimensional state-time space (18 state variables and one switching time) consisting of two legs, two thighs, two arms, two forearms, and a torso. A representative gait can be seen in Figure 1(d). For this model we have taken the sagittal-plane parameters from [25]. There are no knee stops and all links are free to rotate the full 360°. As before, we start with our template configuration and search for singular points. One singular point occurs at $t \approx 0.49$ s. Unlike the compass gait, the switching times for this model do not come in pairs. Then we apply continuation methods to generate the 1-D curve of passive solutions. While we were expecting human-like walking motions, the gaits resemble more of a bipedal bird-like, flamingo gait (we could also claim that this is a useful gait for a person walking downstairs backwards!) Another odd behavior about this gait is that although it has two arms, they only swing in phase with each other. We leave it for future work to extend the nine-link model to a human-like gait using a similar approach to that used for kneed walkers [26]. But even for a complicated walking model, the search space is only 1-D and once the singular points are found we are able to generate periodic motions with ease.

C. *Comparing our approach to the use of virtual holonomic constraints for generating gaits for a three-link walker*

We now examine the three-link model presented in [5] (Figure 1(b)). Details of the model can be found therein. For this biped, $k = 2$ and $\lambda = [\kappa, u]$, where κ is torsional spring constant and u is a constant torque applied at the hip joint. The purpose of adding controls was to induce stable, periodic walking on level ground using virtual holonomic constraints (VHC). The paper also gives a stabilizing controller for the

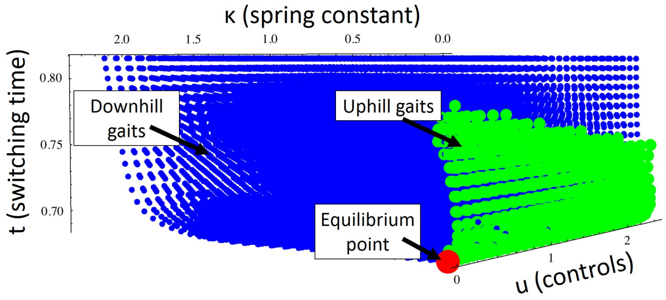


Fig. 6. The 3-D surface of solutions projected onto the switching time (t)-spring constant (κ)-torque (u) plane for the compass-gait biped with torso. The red dot is the equilibrium point that generated all of the fixed points, green dots are uphill gaits, and blue dots are downhill (passive and actuated) gaits.

biped to use, while this paper assumes a stabilizing controller is given. While none of the gaits we found for the three-link biped are open-loop stable, we note that our approach generates a manifold of bipedal gaits that complements the controller in the paper allowing for a decoupling of the gait generation problem from the gait stabilization problem. This is in contrast to the output of the gait generator produced by the VHC function which does not output a periodic gait, but a motion with a normed difference of 10^{-3} between two consecutive post-impact states (the paper refers to the output as a "quasi-periodic" gait). It is up to the controller to further reduce the error. Thus the VHC gait generator is tightly coupled to the controller.

Furthermore, searching for gaits using VHC relies on optimization methods to search for a gait over the paper's state-control space. One of the gaits took about 3 s to find and the gait was found using initial conditions from work on a previous three-link walker. It is not obvious how the VHC gait generation algorithm can scale up to the nine-link humanoid system with the initial guess, coordinating function, and reliance on an external function to make the motion periodic being an integral part to the algorithm's success.

On the other hand, our approach has specific instructions on how to generate periodic gaits. All of the gaits shown in this paper have a normed error between post-impact states of at most 10^{-8}. When searching for gaits for the compass-gait model with torso, the singular points in the state-time subspace occur at $t \approx 0.66$ s and $t \approx 0.69$ s. After locating these switching times, we chose to extend the longer period gait into the state-time-control space and ultimately generated 67,600 fixed points. Figure 6 shows a selection of these points. We used a step size of $\frac{1}{10}$ spanning each of the corresponding direction in the time-controls parameter space of switching time, spring constant, and torque constant. The values for the control parameters had ranges of $t^* \in [0.68, 0.82]$, $\kappa^* \approx \pm 2.47$, and $u^* \approx \pm 2.12$. We ran our code in Mathematica on two cores. The total time in Mathematica took 2,071.6 s at an average of about 33 fixed points per second on a 3.06 GHz Duo Core 2 processor with code running on both cores in parallel. We generated the gaits using the procedure of Algorithm 1 by holding 2 variables fixed in the time-controls subspace and varying the third.

V. Conclusion

Generating periodic walking gaits for an arbitrary biped model is still a challenging problem in the walking community. We present a robust method for generating a family of walking gaits using numerical continuation methods based on the connected component of one (trivial) equilibrium point of an multi-degree-of-freedom, planar, bipedal robot (that of standing still).

Our approach differs from other gait generation algorithms in that we transform the problem of gait generation to generating a connected surface (or level set) of periodic walking gaits. The major advantage is that we give an initial stance for finding nearby walking gaits regardless of the degrees of freedom in the model. We prove that the complexity of the search space only grows with the periodicity of the gait. We also preserve the "natural" dynamics of the hybrid system without resorting to model reduction or splines and easily incorporate control and design parameters when generating periodic walking motions.

As examples, we showed how our algorithm can generate numerous gaits for the compass-gait model, three-link model with legs and torso, and a nine-link humanoid model complete with elbows and knees. Additionally, in our three-link example, we compared our approach to generating gaits to that of using virtual holonomic constraint comparing the strengths and weaknesses in each approach.

Two drawbacks with our approach are that the gaits found are unstable (except for the compass gait) and may not be optimal subject to a user-defined criterion. The choice of controller will impact the stability of a gait. The gaits we found in our examples used zero or a constant torque as input, but as long as one can compute the derivative of the step-to-step map with respect to their controls, then our method is still applicable. However, further extensions are needed to find optimal gaits. Away from a singular point, fixed points form manifolds and finding optimal points on a manifold is a well-studied problem.

For future work, we plan to show how to extend our approach to 3-D and expand on how design parameters can easily be incorporated by making simple modifications to the Recursive Euler-Newton and Composite Rigid Body algorithms thus addressing the issue of having to take derivatives of a trajectory with respect to the parameters by automating the process algorithmically.

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