The Passive Dynamics of Walking and Brachiating Robots: Results on the Topology and Stability of Passive Gaits

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Simple walking models, like the compass-gait model, have yielded useful insight into the basic mechanics of walking. A similar model serves as a template for brachiation. With the ability of a two-link robot to walk and swing, we explore the multi-locomotion capability of a generalized two-link model with potential footholds at any location. We focus on the connected components of passive gaits in a five-dimensional state-time space. Our main results are: (1) a walking gait and a brachiating gait cannot be in the same connected component and (2) the stability of a gait depends on whether impacts are state-based (e.g., footfall in a biped walker) or time-based (e.g., time between clamping brachiator hands to a wall). For the same connected component of gaits, the different impact types result in different bifurcations.

Keywords: brachiation; walking; compass-gait; simplest walking model

1. Introduction

We study the passive hybrid dynamics of planar, single-joint, two-link walking and brachiating robot models. These simplified models of walking and brachiating have been studied extensively in the past.[1–4] These locomotors are modeled as a hybrid system, where two continuous swing motions are separated by an impact with a surface causing a discontinuous jump in velocity.

We generalize the two-link walking models[1,3] and brachiator models[2,5] to a single model that can both walk and brachiate. This allows us to study robots like our wall-climbing Gibbot robot,[4] which can achieve a "foothold" anywhere on a vertical plane. Since the "feet" of the generalized model can achieve a foothold at any location on a vertical plane, we can study two impact strategies: switching the stance foot when (1) it "collides" with a fixed virtual slope (state-based impacts) or (2) a set period of time
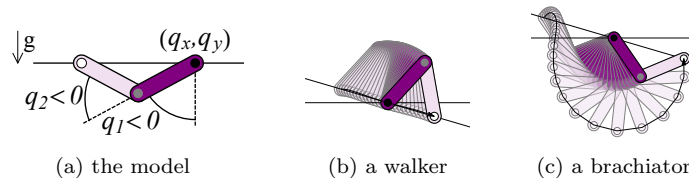
Fig. 1.   An example of (a) the two-link model (b) walking and (c) brachiating on the same downhill slope. The swing leg is the lightly shaded link.

has elapsed (time-based impacts). Figure 1 shows a simple two-link model walking and brachiating on the same downhill slope.

In the past, simple two-link models have successfully guided the design of several walking and brachiating robots.[5–8] Yet, with the exception of work by Fukuda et al.,[9] walking and brachiating robots are treated separately because of their distinct environments. Given our generalized model, could walking and brachiating gaits actually be considered examples of the same type of gait? Furthermore, the same gait can be achieved by either state-based or time-based impact. But is there a difference in stability depending on the type of impact?

We address these two questions by studying the connected components and stability of gaits in our generalized model. We define a gait as a period-one fixed point of the hybrid dynamics in a five-dimensional state-time space. For a period-one fixed point, the state corresponds to the robot's post-impact state at the beginning of a new swing and time corresponds to the period of time between impacts. Gaits in the same connected component have a path connecting them.

Definition 1. A *path* between two gaits a and b is a continuous function f from the unit interval $[0, 1]$ to the set of gaits in $\mathbb{R}^5$ such that $f(0) = a$ and $f(1) = b$.

If a path exists between two gaits, then it is possible to continuously deform one gait into the other. We give an example in Section 3.

Similar to the terminology used in the walking community, we refer to the link that is in contact with the surface as the stance leg and the other link as the swing leg. After an impact the roles of the two links switch. For motions of the swing leg relative to the stance leg where the net displacement does not exceed one revolution during a step, the net displacement can only have two possible values. These values map the swing leg trajectory to a net motion where the links cross once or not at all. We define walking and brachiating gaits based on this notion of links crossing.

Definition 2. The links *cross* when the net angular displacement of the swing leg relative to the stance leg does not equal the difference between the final and initial angles of the swing leg relative to the stance leg. For a *walking* gait, the links cross. For a *brachiating* gait, the links do not cross.

We return to this definition in Section 3. Our contributions are

(1) **Walking and brachiating fixed points are not in the same connected component of gaits.** We show that walking and brachiating gaits are two disjoint sets. We explore a connected component in each set through numerical simulation and show that both gaits can passively locomote above or below a fixed slope. For example, we show that it is possible to continuously deform a walking gait that walks on ground to a gait that passively "walks" on an inclined ceiling.

(2) **The impact strategy affects the stability of a gait.** We find that state-based switching leads to more stable gaits than time-based switching. When gaits become unstable, state-based switching leads to period-doubling bifurcations (stable period-$2n$ gaits, $n \geq 1$), and time-based switching leads to Neimark-Sacker bifurcations (n-periodic or quasiperiodic gaits, $n > 1$).

We define the hybrid dynamics of the system in Section 2. In Sections 3 and Section 4 we show (1) and (2), respectively. We conclude in Section 5.

2. The Hybrid Dynamics

The physical parameters of each link of the robot are identical to each other, which allows us to define a gait in half a swing of the robot. The configuration vector (Figure 1(a)) of the robot is $q' = [q_x, q_y, q_1, q_2]^T$, where (q_x, q_y) are the x-y coordinates of the stance leg in a world frame, q_1 is the angle of the stance leg from the vertical, and q_2 is the angle of the swing leg relative to the stance leg. The pivot point (q_x, q_y) remains fixed throughout the swing motion. For convenience, during the swing we use a reduced configuration vector $q = [q_1, q_2]^T$ with state vector $x = [q^T, \dot{q}^T]^T$. We define the flow of the continuous dynamics of the double pendulum as

$$x(x_0, t) = x_0 + \int_0^t F(x(s)) \, ds,$$

where x_0 is the initial state of the robot at time $s = 0$, t is the impact time, and $F(x(s)) = [\dot{q}(s)^T, \ddot{q}(s)^T]^T$.

When the robot impacts, it undergoes an instantaneous, plastic impact. The configuration of the robot does not change, but its velocity does. The

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impact map H relates the pre-impact state to the post-impact state of the robot at the time of impact and is defined as

$$H(x) = \begin{bmatrix} A & 0 \\ 0 & P(q) \end{bmatrix} x + \begin{bmatrix} b^T & 0 & 0 \end{bmatrix}^T,$$

where $A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ and $b = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \in \mathbb{R}^2$ flip the coordinate system, and $P(q) \in \mathbb{R}^{2 \times 2}$ relates the pre-impact velocity $\dot{q}^-$ to the post-impact velocity $\dot{q}^+$ such that $\dot{q}^+ = P(q^-)\dot{q}^-$. A detailed derivation of the equations and parameters used for our two-link model can be found in a previous paper.[4]

In sum, the hybrid system is written as a discrete map $G \colon \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$

$$x_1 = H(x(x_0, t)) = G(x_0, t),$$

where x_0 is the initial state, $x(x_0, t)$ is the pre-impact state, and x_1 is the final state after the impact at time t. A "gait" is a period-one fixed point (x_0^*, t^*) satisfying $x_0^* = G(x_0^*, t^*)$. The fixed point is stable if the maximum eigenvalue of $\frac{\partial G}{\partial x_0}(x_0^*, t^*)$ is inside the unit circle.

3. Connected components in the state-time space

In our search for gaits, we allow the state and impact time to vary. In this five-dimensional state-time space, we define the set of all gaits as

$$X = \left\{ (x_0, t) \in \mathbb{R}^5 \mid G(x_0, t) - x_0 = 0 \right\},$$

where x_0, t, and G are defined in Section 2. Let $S = (X, \mathcal{O} \cap X)$ be a subspace topology of the standard topology $T = (\mathbb{R}^5, \mathcal{O})$, where $\mathcal{O}$ is the collection of open sets in $\mathbb{R}^5$. By the implicit function theorem, if the Jacobian matrix

$$J(x_0^*, t^*) = \begin{bmatrix} \dfrac{\partial G}{\partial x_0} & \dfrac{\partial G}{\partial t} \end{bmatrix} (x_0^*, t^*) - \begin{bmatrix} \dfrac{\partial x_0}{\partial x_0} & \dfrac{\partial x_0}{\partial t} \end{bmatrix}$$

has maximal rank at a fixed point (x_0^*, t^*), then nearby solutions form a 1-D curve passing through (x_0^*, t^*) in the state-time space. If $J(x_0^*, t^*)$ does not have maximal rank, then locally the nature of the connected component requires further analysis. It is at these fixed points where multiple 1-D curves can intersect. We trace the connected components of X using an Euler-Newton numerical continuation method.[10] There are many connected components in the 5-D state-time space. Figure 2 shows parts of two connected components projected onto a 2-D subspace of the state-time space. The connected component of walking gaits in Figure 2(a) is comprised of three 1-D curves glued together at two fixed points ($[\pi, \pi, 0, 0]^T, 1.59$) and
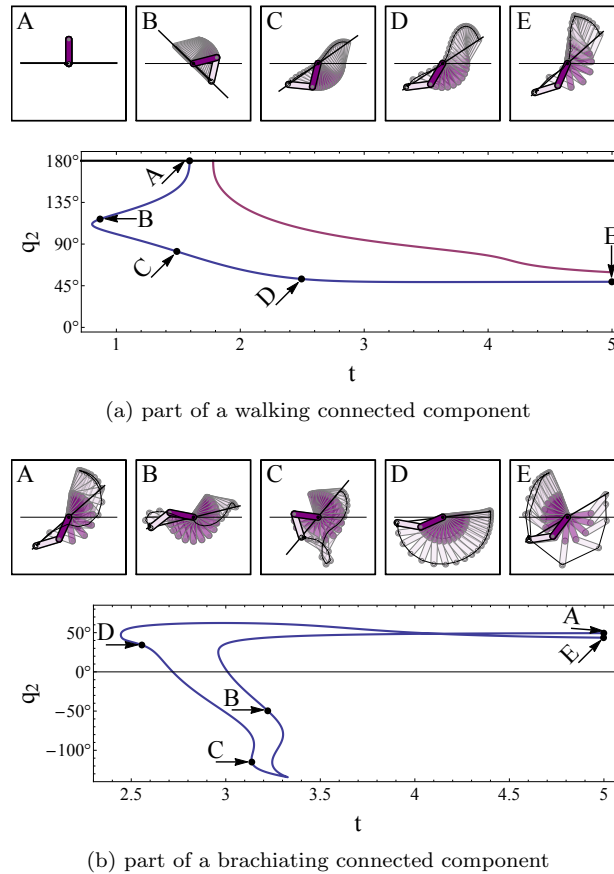
(a) part of a walking connected component

(b) part of a brachiating connected component

Fig. 2.   Parts of two connected components projected onto the q_2-t plane. Every point is a gait in the state-time space. The cartoon animations are the swing motions at points A-E. The walking connected component has an infinite number of solutions[3] branching off the curve at $q_2 = 180°$, while the brachiating curve does not have branching solutions.

$([\pi, \pi, 0, 0]^T, 1.78)$, where $J(x_0^*, t^*)$ is not maximal rank. The black, straight-line curve at $q_2 = 180°$ consists of the state $[\pi, \pi, 0, 0]^T$ for all switching times t^*. The state is an equilibrium point of the double pendulum dynamics. For the connected component of brachiating gaits in Figure 2(b), $J(x_0^*, t^*)$ evaluated at every fixed point along the curve has maximal rank. Because of this, the curve has a unique arc-length parameterization.[11] There are other 1-D solution curves of brachiating gaits (see, e.g., Figure 4), but these curves are not connected to each other.

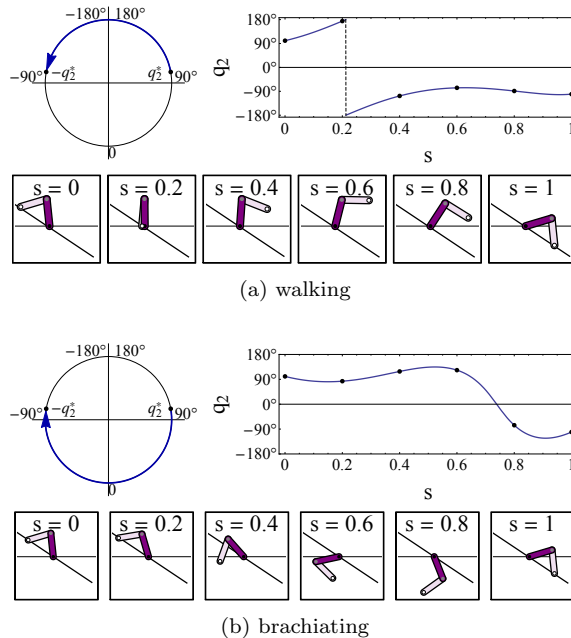In the Introduction, we defined walking and brachiating based on

(a) walking

(b) brachiating

Fig. 3.   The trajectory of the swing leg of two gaits with the same initial configuration $q \approx (-174.1°, 100.6°)$ are shown. The arrow on the circle corresponds to the net displacement of the swing leg relative to the stance leg, the plot is the swing leg's trajectory scaled to each gait's switching time, and the animations are snapshots of the robot's trajectory over time. The net displacement of the walking gait (a) crosses 180° (i.e., the links cross), while the brachiating gait (b) crosses zero (i.e., the links did not cross).

whether the links cross or not. For a trajectory to be a fixed point, the angle of the swing leg must equal $q_2(t) = -q_2(0) = -q_2^*$ prior to impact. This puts a constraint on the net angular displacement of the swing leg

$$\Delta q_2 = \int_0^t \dot{q}_2(s)ds = 2\pi k - 2q_2^*,$$

where k is an integer. For $|\Delta q_2| \leq 2\pi$, the net angular displacement has the intuitive meaning of the links crossing or not crossing (Figure 3). If $q_2^* \in [-\pi, \pi]$,[a] then $k \in \{-1, 0, 1\}$, where $k = 0$ is a brachiating gait and $|k| = 1$ is a walking gait. We have now partitioned the set of fixed points into two sets. Let B and W be the set of fixed points with values of $|k|$ equal to 0 and 1, respectively. By construction, we have that $B \cup W = X$. If there is a continuous path between fixed points in B and W, then there

[a]Removing $-\pi$ or π excludes brachiating either counterclockwise or clockwise at $q_2^* = \pi$.

must exist a fixed point on the path that is in both sets. Such a fixed point cannot exist, because this would mean that the same state has two different trajectories—one where the links cross and another where they do not. This cannot be true as solutions to the differential equation of the double pendulum are unique. Hence, $B \cap W = \emptyset$ and no path can exist between gaits in these sets.

While a continuous path cannot exist between a walking gait and a brachiating gait, connected components in both B and W connect gaits that locomote above and below the surface. Figure 2(a) shows a path in W that continuously deforms a walking gait starting at the gait labeled B to what is often referred to as "over-hand"[9] brachiation (gait C). Topologically, this gait is equivalent to a gait that "walks" below the surface.
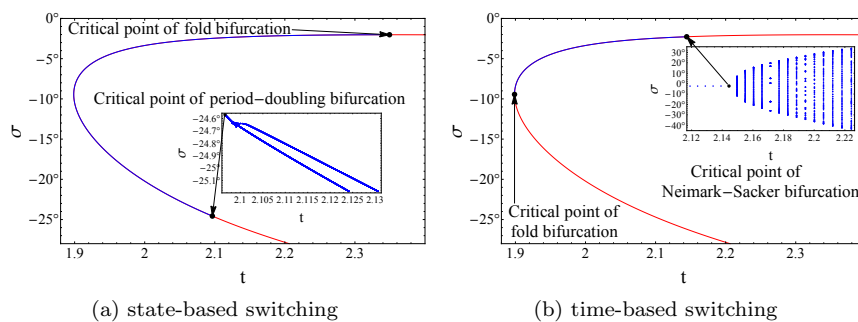
4. Stability and Bifurcations

Fig. 4.   Stability of gaits on a brachiating solution curve under (a) state-based and (b) time-based impacts. The impacts can occur at a fixed slope σ or switching time t. The blue segments of the curve are stable fixed points and the red are unstable. The types of bifurcation are also highlighted. The insets show the higher period gaits that result.

The switching time t plays an interesting role in our model. If we treat it as a free parameter, then we can impact whenever we want. If instead the robot impacts when it returns back to its initial slope σ, then $t = t(x_0)$ is dependent on the state. Figure 4 shows the stability of a 1-D curve of brachiating gaits under the two switching strategies as we move along the curve (this curve does not intersect the curve in Figure 2(b), which does not have stable time-based gaits). In the example of Figure 4, the set of stable time-based switching gaits is a subset of the set of stable state-based switching gaits. In our experience, all gaits that are stable under time-based switching are also stable under state-based switching, but the converse is not true. While both switching strategies give rise to fold bifurcations, we have also

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observed period-doubling bifurcations in the case of state-based switching and Neimark-Sacker bifurcations[12] in the case of time-based switching.

5. Conclusion

We have presented results on the connected components and stability of gaits for a two-link robot capable of brachiating and walking using a generalized two-link model. We have shown that walking and brachiating are distinct gaits, but they are not differentiated by moving above or below a slope. We have also shown that the impact strategy affects the stability of a gait, where state-based impacts are more stable than time-based impacts.

For future work, we plan to extend our model to powered gaits in a state-control space that includes control parameters for an actuator at the joint. Do connected components remain disconnected in this higher-dimensional state-control space or are they part of the same connected component that appears disconnected when projected onto the 5-D state-time space of passive gaits studied in this paper?

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